Analytical Blowup Solutions to the 4-dimensional Pressureless Navier-Stokes-Poisson Equations with Density-dependent Viscosity

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Abstract

We study the 4-dimensional pressureless Navier–Stokes-Poisson equations with density-dependent viscosity. The analytical solutions with arbitrary time blowup, in radial symmetry, are constructed in this paper.

1 Introduction

The evolution of a self-gravitating fluid can be formulated by the Navier-Stokes-Poisson equations of the following form:

$$\begin{cases} \rho_t + \nabla \bullet (\rho u) = 0, \\ (\rho u)_t + \nabla \bullet (\rho u \otimes u) + \nabla P = -\rho \nabla \Phi + vis(\rho, u), \\ \Delta \Phi(t, x) = \alpha(N)\rho, \end{cases} \tag{1}$$

where $\alpha(N)$ is a constant related to the unit ball in R^N: $\alpha(1) = 2$; $\alpha(2) = 2\pi$ and For $N \geq 3$,

$$\alpha(N) = N(N-2)V(N) = N(N-2)\frac{\pi^{N/2}}{\Gamma(N/2+1)}, \tag{2}$$

where $V(N)$ is the volume of the unit ball in R^N and Γ is a Gamma function. And as usual, $\rho = \rho(t, x)$ and $u = u(t, x) \in \mathbf{R}^N$ are the density, the velocity respectively. $P = P(\rho)$ is the pressure.

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In the above system, the self-gravitational potential field $\Phi = \Phi(t,x)$ is determined by the density ρ through the Poisson equation.

And $vis(\rho,u)$ is the viscosity function:

$$vis(\rho,u) = \bigtriangledown(\mu(\rho) \bigtriangledown \bullet u). \tag{3}$$

Here we under a common assumption for:

$$\mu(\rho) \doteq \kappa\rho^{\theta} \tag{4}$$

and κ and $\theta \geq 0$ are the constants. In particular, when $\theta = 0$, it returns the expression for the u dependent only viscosity function:

$$vis(\rho,u) = \kappa\Delta u. \tag{5}$$

And the vector Laplacian in $u(t,r)$ can be expressed:

$$\Delta u = u_{rr} + \frac{N-1}{r}u_r - \frac{N-1}{r^2}u. \tag{6}$$

The equations $(1)_1$ and $(1)_2$ ($vis(\rho,u) \neq 0$) are the compressible Navier-Stokes equations with forcing term. The equation $(1)_3$ is the Poisson equation through which the gravitational potential is determined by the density distribution of the density itself. Thus, we call the system (1) the Navier–Stokes-Poisson equations.

Here, if the $vis(\rho,u) = 0$, the system is called the Euler-Poisson equations. In this case, the equations can be viewed as a prefect gas model. For $N = 3$, (1) is a classical (nonrelativistic) description of a galaxy, in astrophysics. See [2], [7] for a detail about the system.

$P = P(\rho)$ is the pressure. The γ-law can be applied on the pressure $P(\rho)$, i.e.

$$P(\rho) = K\rho^{\gamma} \doteq \frac{\rho^{\gamma}}{\gamma}, \tag{7}$$

which is a commonly the hypothesis. The constant $\gamma = c_P/c_v \geq 1$, where c_P, c_v are the specific heats per unit mass under constant pressure and constant volume respectively, is the ratio of the specific heats, that is, the adiabatic exponent in (7). In particular, the fluid is called isothermal if $\gamma = 1$. With $K = 0$, we call the system is pressureless.

For the 3-dimensional case, we are interested in the hydrostatic equilibrium specified by $u = 0$. According to [2], the ratio between the core density $\rho(0)$ and the mean density $\bar{\rho}$ for $6/5 < \gamma < 2$ is given by

$$\frac{\bar{\rho}}{\rho(0)} = \left(\frac{-3}{z}\dot{y}(z)\right)_{z=z_0} \tag{8}$$

where y is the solution of the Lane-Emden equation with $n = 1/(\gamma - 1)$,

$$\ddot{y}(z) + \frac{2}{z}\dot{y}(z) + y(z)^n = 0,\ y(0) = \alpha > 0,\ \dot{y}(0) = 0,\ n = \frac{1}{\gamma-1}, \tag{9}$$

and z_0 is the first zero of $y(z_0) = 0$. We can solve the Lane-Emden equation analytically for

$$y_{anal}(z) \doteq \begin{cases} 1 - \frac{1}{6}z^2, & n = 0; \\ \dfrac{\sin z}{z}, & n = 1; \\ \dfrac{1}{\sqrt{1 + z^2/3}}, & n = 5, \end{cases} \tag{10}$$

and for the other values, only numerical values can be obtained. It can be shown that for $n < 5$, the radius of polytropic models is finite; for $n \geq 5$, the radius is infinite.

Gambin [4] and Bezard [1] obtained the existence results about the explicitly stationary solution $(u = 0)$ for $\gamma = 6/5$ in Euler-Poisson equations:

$$\rho = \left(\frac{3KA^2}{2\pi}\right)^{5/4} \left(1 + A^2 r^2\right)^{-5/2}, \tag{11}$$

where A is constant.

The Poisson equation $(1)_3$ can be solved as

$$\Phi(t, x) = \int_{R^N} G(x - y)\rho(t, y)dy, \tag{12}$$

where G is the Green's function for the Poisson equation in the N-dimensional spaces defined by

$$G(x) \doteq \begin{cases} |x|, & N = 1; \\ \log |x|, & N = 2; \\ \dfrac{-1}{|x|^{N-2}}, & N \geq 3. \end{cases} \tag{13}$$

In the following, we always seek solutions in radial symmetry. Thus, the Poisson equation $(1)_3$ is transformed to

$$r^{N-1}\Phi_{rr}\left(t, x\right) + \left(N - 1\right) r^{N-2}\Phi_r = \alpha\left(N\right) \rho r^{N-1}, \tag{14}$$

$$\Phi_r = \frac{\alpha\left(N\right)}{r^{N-1}} \int_0^r \rho(t, s)s^{N-1}ds. \tag{15}$$

Definition 1 (Blowup) *We say a solution blows up if one of the following conditions is satisfied:*
(1)The solution becomes infinitely large at some point x and some finite time T_0;
(2)The derivative of the solution becomes infinitely large at some point x and some finite time T_0.

In this paper, we concern about blowup solutions for the 4-dimensional pressureless Navier–Stokes-Poisson equations with the density-dependent viscosity. And our aim is to construct a family of such blowup solutions.

Historically in astrophysics, Goldreich and Weber [5] constructed the analytical blowup solution (collapsing) of the 3-dimensional Euler-Poisson equations for $\gamma = 4/3$ for the non-rotating gas spheres. After that, Makino [7] obtained the rigorously mathematical proof of the existence of such kind of blowup solutions. And in [3], we find the extension of the above blowup solutions to the case . In [8], the solutions with a from is rewritten as

For $N\geq 3$ and $\gamma=(2N-2)/N$,

$$\begin{cases} \rho(t,r)=\begin{cases}\dfrac{1}{a(t)^N}y(\frac{r}{a(t)})^{N/(N-2)}, \text{ for } r<a(t)Z_\mu; \\ 0, \text{ for } a(t)Z_\mu\leq r.\end{cases}, \ u(t,r)=\dfrac{\dot{a}(t)}{a(t)}r, \\ \ddot{a}(t)=-\dfrac{\lambda}{a(t)^{N-1}}, \ a(0)=a_0>0, \ \dot{a}(0)=a_1, \\ \ddot{y}(z)+\dfrac{N-1}{z}\dot{y}(z)+\dfrac{\alpha(N)}{(2N-2)K}y(z)^{N/(N-2)}=\mu, \ y(0)=\alpha>0, \ \dot{y}(0)=0, \end{cases} \tag{16}$$

where $\mu=[N(N-2)\lambda]/(2N-2)K$ and the finite Z_μ is the first zero of $y(z)$;

For $N=2$ and $\gamma=1$,

$$\begin{cases} \rho(t,r)=\dfrac{1}{a(t)^2}e^{y(r/a(t))}, \ u(t,r)=\dfrac{\dot{a}(t)}{a(t)}r; \\ \ddot{a}(t)=-\dfrac{\lambda}{a(t)}, \ a(0)=a_0>0, \ \dot{a}(0)=a_1; \\ \ddot{y}(z)+\dfrac{1}{z}\dot{y}(z)+\dfrac{2\pi}{K}e^{y(z)}=\mu, \ y(0)=\alpha, \ \dot{y}(0)=0, \end{cases} \tag{17}$$

where $K>0$, $\mu=2\lambda/K$ with a sufficiently small λ and α are constants.

For the construction of special analytical solutions to the Navier-Stokes equations in R^N or the Navier-Stokes-Poisson equations in R^3 without pressure with $\theta=1$, readers may refer Yuen's recent results in [10], [11] respectively.

In this article, the analytical blowup solutions are constructed in the pressureless Euler-Poisson equations with density-dependent viscosity in R^4 with $\theta=5/4$ in radial symmetry:

$$\begin{cases} \rho_t+u\rho_r+\rho u_r+\dfrac{3}{r}\rho u=0, \\ \rho\left(u_t+uu_r\right)=-\dfrac{\alpha(4)\rho}{r^3}\displaystyle\int_0^r\rho(t,s)s^3ds+[\kappa\rho^{5/4}]_r u_r+(\kappa\rho^{5/4})(u_{rr}+\dfrac{3}{r}u_r-\dfrac{3}{r^2}u), \end{cases} \tag{18}$$

in the form of the following theorem.

Theorem 2 *For the* 4*-dimensional pressureless Navier–Stokes-Poisson equations with* $\theta=5/4$*, in radial symmetry, (18), there exists a family of blowup solutions,*

$$\begin{cases} \rho(t,r)=\dfrac{1}{(T-Ct)^4}y(\frac{r}{T-Ct})^4, \ u(t,r)=\dfrac{-C}{T-Ct}r; \\ \ddot{y}(z)+\dfrac{3}{z}\dot{y}(z)+\dfrac{\alpha(4)}{\kappa C}y(z)^4=0, \ y(0)=\alpha, \ \dot{y}(0)=0, \end{cases} \tag{19}$$

where $T>0$*,* $\kappa>0$*,* $C>0$ *and* α *are constants.*

And the solutions blow up in the finite time T/C*.*

2 Separable Blowup Solutions

Before presenting the proof of Theorem 2, we prepare some lemmas. First, we obtain the solutions for the continuity equation of mass in radial symmetry $(18)_1$.

Lemma 3 *For the 4-dimensional conservation of mass in radial symmetry*

$$\rho_t + u\rho_r + \rho u_r + \frac{3}{r}\rho u = 0, \tag{20}$$

there exist solutions,

$$\rho(t,r) = \frac{1}{(T-Ct)^4} y(\frac{r}{T-Ct})^4,\ u(t,r) = \frac{-C}{T-Ct} r, \tag{21}$$

where T and C are positive constants.

Proof. We just plug (21) into (20). Then

$$\begin{aligned}
&\rho_t + \rho_r u + \rho u_r + \frac{3}{r}\rho u \\
&= \frac{(-4)(-C)y(\frac{r}{T-Ct})^4}{(T-Ct)^4} + \frac{4y(\frac{r}{T-Ct})^3 \dot{y}(\frac{r}{T-Ct})}{(T-Ct)^4} \frac{r(-1)(-C)}{(T-Ct)^2} \\
&+ \frac{4y(\frac{r}{T-Ct})^3 \dot{y}(\frac{r}{T-Ct})}{(T-Ct)^4} \frac{1}{T-Ct} \frac{(-C)}{T-Ct} r + \frac{y(\frac{r}{T-Ct})^4}{(T-Ct)^4} \frac{(-C)}{T-Ct} + \frac{3}{r} \frac{y(\frac{r}{T-Ct})^4}{(T-Ct)^3} \frac{(-C)}{T-Ct} r \\
&= \frac{4Cy(\frac{r}{T-Ct})^4}{(T-Ct)^4} + \frac{4Cy(\frac{r}{T-Ct})^3 \dot{y}(\frac{r}{T-Ct})}{(T-Ct)^6} r \\
&- \frac{4Cy(\frac{r}{T-Ct})^3 \dot{y}(r/(T-Ct))}{(T-Ct)^6} r - \frac{Cy(\frac{r}{T-Ct})^4}{(T-Ct)^4} - \frac{3Cy(\frac{r}{T-Ct})^4}{(T-Ct)^4} \\
&= 0.
\end{aligned}$$

The proof is completed. ■

Besides, we need the lemma for stating the property of the function $y(z)$. The similar lemma was already given in Lemmas 2.1, [3], by the fixed point theorem. The proof is similar, the proof may be skipped here.

Lemma 4 *For the ordinary differential equation,*

$$\begin{cases} \ddot{y}(z) + \frac{3}{z}\dot{y}(z) + \frac{\alpha(4)}{5C\kappa} y(z)^4 = 0, \\ y(0) = \alpha > 0,\ \dot{y}(0) = 0, \end{cases} \tag{22}$$

where $\alpha(4)$, C and κ are positive constants,
has a solution $y(z) \in C^2$ provided that $y(z) \subset [\alpha, 0]$.

Here we are already to give the proof of Theorem 2.

Proof of Theorem 2. From Lemma 3, it is clear for that (19) satisfy $(18)_1$. For the momentum equation $(18)_2$, we get,

$$\rho(u_t+uu_r)+\frac{\alpha(4)\rho}{r^3}\int_0^r \rho(t,s)s^3ds-[\kappa\rho^{5/4}]_r u_r-\kappa\rho^{5/4}(u_{rr}+\frac{3}{r}u_r-\frac{3}{r^2}u) \tag{23}$$

$$=\rho\left[\frac{(-C)(-1)(-C)}{(T-Ct)^2}r+\frac{(-C)}{T-Ct}r\cdot\frac{(-C)}{T-Ct}\right]+\frac{\alpha(4)\rho}{r^3}\int_0^r \frac{y(\frac{s}{T-Ct})^4}{(T-Ct)^4}s^3ds \tag{24}$$

$$-\left[\kappa\frac{1}{(T-Ct)^4}y\left(\frac{r}{T-Ct}\right)^4\right]_r^{5/4}\frac{(-C)}{T-Ct}-0$$

$$=\frac{\alpha(4)\rho}{r^3}\int_0^r \frac{y(\frac{s}{T-ct})^4}{(T-Ct)^4}s^3ds-\frac{5}{4}\kappa\left[\frac{1}{(T-Ct)^4}y\left(\frac{r}{T-Ct}\right)^4\right]^{1/4}\frac{4y\left(\frac{r}{T-Ct}\right)^3\dot{y}\left(\frac{r}{T-Ct}\right)}{(T-Ct)^4}\frac{1}{T-Ct}\frac{(-C)}{T-Ct} \tag{25}$$

$$=\frac{\alpha(4)\rho}{r^3}\int_0^r \frac{y(\frac{s}{T-Ct})^4}{(T-Ct)^4}s^3ds+5C\kappa\frac{y\left(\frac{r}{T-Ct}\right)}{T-Ct}\frac{y\left(\frac{r}{T-Ct}\right)^3\dot{y}\left(\frac{r}{T-Ct}\right)}{(T-Ct)^4}\frac{1}{(T-Ct)^2} \tag{26}$$

$$=\frac{\alpha(4)\rho}{r^3}\int_0^r \frac{y(\frac{s}{T-Ct})^4}{(T-Ct)^4}s^3ds+\frac{5C\kappa\rho}{(T-Ct)^3}\dot{y}\left(\frac{r}{T-Ct}\right) \tag{27}$$

$$=\frac{\rho}{(T-Ct)^3}\left[5C\kappa\dot{y}(\frac{r}{T-Ct})+\frac{\alpha(4)}{r^3(T-Ct)}\int_0^r y(\frac{s}{T-Ct})^4s^3ds\right] \tag{28}$$

$$=\frac{\rho}{(T-Ct)^3}\left[5C\kappa\dot{y}(\frac{r}{T-Ct})+\frac{\alpha(4)}{(\frac{r}{T-Ct})^3}\int_0^{r/(T-Ct)} y(s)^4s^3ds\right] \tag{29}$$

$$=\frac{\rho}{(T-Ct)^3}Q\left(\frac{r}{T-Ct}\right). \tag{30}$$

And denote

$$Q(\frac{r}{T-Ct})\doteq Q(z)=5C\kappa\dot{y}(z)+\frac{\alpha(4)}{z^3}\int_0^z y(s)^4s^3ds. \tag{31}$$

Differentiate $Q(z)$ with respect to z,

$$\dot{Q}(z) \tag{32}$$

$$=5C\kappa\ddot{y}(z)+\alpha(4)y(z)^4+\frac{(-3)\alpha(4)}{z^4}\int_0^z y(s)^4s^3ds \tag{33}$$

$$=-\frac{3}{z}\cdot 5C\kappa\dot{y}(z)+\frac{(-3)}{z}\cdot\frac{\alpha(4)}{z^3}\int_0^z y(s)^4s^3ds \tag{34}$$

$$=\frac{-3}{z}Q(z), \tag{35}$$

where the above result is due to the fact that we choose the following ordinary differential equation,

$$\begin{cases} \ddot{y}(z) + \dfrac{3}{z}\dot{y}(z) + \dfrac{\alpha(4)}{5C\kappa}y(z)^4 = 0. \\ \quad y(0) = \alpha > 0,\ \dot{y}(0)= 0. \end{cases} \tag{36}$$

With $Q(0) = 0$, this implies that $Q(z) = 0$. Thus, the momentum equation $(18)_2$ is satisfied. Now we are able to show that the family of the solutions blow up, in the finite time T/C. This completes the proof. ■

The statement about the blowup rate will be immediately followed:

Corollary 5 *The blowup rate of the solution (19) is*

$$\lim_{t\to T/C^-} \rho(t,0)(T-Ct)^4 \geq O(1). \tag{37}$$

Given that the sign of the constant C in (17) is changed to be negative, the below corollary is clearly shown.

Corollary 6 *For the* 4*-dimensional pressureless Navier–Stokes-Poisson equations, with* $\theta = 5/4$*, in radial symmetry, (18), there exists a family of solutions,*

$$\begin{cases} \rho(t,r) = \dfrac{1}{(T-Ct)^4}y\left(\frac{r}{T-Ct}\right)^4,\ u(t,r) = \dfrac{-C}{T-Ct}r; \\ \ddot{y}(z)+\dfrac{3}{z}\dot{y}(z)+\dfrac{\alpha(4)}{5C\kappa}y(z)^4 = 0,\ y(0) = \alpha > 0,\ \dot{y}(0) = 0, \end{cases} \tag{38}$$

where $T > 0$*,* $\kappa > 0$*,* $C < 0$ *and* α *are constants.*

Remark 7 *Besides, if we consider the* 4*-dimensional Navier-Stokes equations with the repulsive force in radial symmetry,*

$$\begin{cases} \rho_t + u\rho_r + \rho u_r + \dfrac{3}{r}\rho u = 0, \\ \qquad \rho\left(u_t + uu_r\right) = +\dfrac{\alpha(4)\rho}{r^3}\int_0^r \rho(t,s)s^3 ds + [\kappa\rho^{5/4}]u_r + (\kappa\rho^{5/4})(u_{rr} + \dfrac{3}{r}u_r - \dfrac{3}{r^2}u), \end{cases} \tag{39}$$

the special solutions are:

$$\begin{cases} \rho(t,r) = \dfrac{1}{(T-Ct)^4}y\left(\frac{r}{T-Ct}\right)^4,\ u(t,r) = \dfrac{-C}{T-Ct}r \\ \ddot{y}(z)+\dfrac{3}{z}\dot{y}(z) - \frac{\alpha(4)}{5C\kappa}y\left(z\right)^4 = 0,\ y(0) = \alpha,\ \dot{y}(0) = 0. \end{cases} \tag{40}$$

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